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Research Article

ϵ-Duality Theorems for Convex Semidefinite Optimization Problems with Conic Constraints

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A convex semidefinite optimization problem with a conic constraint is considered. We formulate a Wolfe-type dual problem for the problem for its ϵ-approximate solutions, and then we prove ϵ-weak duality theorem and ϵ-strong duality theorem which hold between the problem and its Wolfe type dual problem. Moreover, we give an example illustrating the duality theorems.

1. Introduction

Convex semidefinite optimization problem is to optimize an objective convex function over a linear matrix inequality. When the objective function is linear and the corresponding matrices are diagonal, this problem becomes a linear optimization problem.

For convex semidefinite optimization problem, Lagrangean duality without constraint qualification [1, 2], complete dual characterization conditions of solutions [1, 3, 4], saddle point theorems [5], and characterizations of optimal solution sets [6, 7] have been investigated.

To get the ϵ-approximate solution, many authors have established ϵ-optimality conditions, ϵ-saddle point theorems and ϵ-duality theorems for several kinds of optimization problems [1, 8–16].

Recently, Jeyakumar and Glover [11] gave ϵ-optimality conditions for convex optimization problems, which hold without any constraint qualification. Yokoyama and Shiraishi [16] gave a special case of convex optimization problem which satisfies ϵ-optimality conditions. Kim and Lee [12] proved sequential ϵ-saddle point theorems and ϵ-duality theorems for convex semidefinite optimization problems which have not conic constraints.

The purpose of this paper is to extend the ϵ-duality theorems by Kim and Lee [12] to convex semidefinite optimization problems with conic constraints. We formulate a Wolfe type dual problem for the problem for its ϵ-approximate solutions, and then prove

ϵ-weak duality theorem and ϵ-strong duality theorem for the problem and its Wolfe type dual problem, which hold under a weakened constraint qualification. Moreover, we give an example illustrating the duality theorems.

2. Preliminaries

Consider the following convex semidefinite optimization problem:

$$\begin{aligned} &\text{(SDP)} \quad \text{Minimize} \quad f(x), \\ &\qquad\quad \text{subject to} \quad F_0 + \sum_{i=1}^{m} x_i F_i \succeq 0, \quad (x_1, x_2, \dots, x_m) \in C, \end{aligned} \tag{2.1}$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a convex function, C is a closed convex cone of $\mathbb{R}^m$, and for $i = 0, 1, \dots, m, F_i \in S_n$, where S_n is the space of $n \times n$ real symmetric matrices. The space S_n is partially ordered by the Löwner order, that is, for $M, N \in S_n, M \succeq N$ if and only if $M - N$ is positive semidefinite. The inner product in S_n is defined by $(M, N) = \mathrm{Tr}[MN]$, where $\mathrm{Tr}[\cdot]$ is the trace operation.

Let $S := \{M \in S_n \mid M \succeq 0\}$. Then S is self-dual, that is,

$$S^+ := \{\theta \in S_n \mid (\theta, Z) \geqq 0, \text{ for any } Z \in S\} = S. \tag{2.2}$$

Let $F(x) := F_0 + \sum_{i=1}^{m} x_i F_i$, $\widehat{F}(x) := \sum_{i=1}^{m} x_i F_i$, $x = (x_1, \dots, x_m) \in \mathbb{R}^m$. Then $\widehat{F}$ is a linear operator from $\mathbb{R}^m$ to S_n and its dual is defined by

$$\widehat{F}^*(Z) = (\mathrm{Tr}[F_1 Z], \dots, \mathrm{Tr}[F_m Z]), \tag{2.3}$$

for any $Z \in S_n$. Clearly, $A := \{x \in C \mid F(x) \in S\}$ is the feasible set of SDP.

Definition 2.1. Let $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a convex function.

(1) The subdifferential of g at $a \in \operatorname{dom} g$, where $\operatorname{dom} g = \{x \in \mathbb{R}^n \mid g(x) < +\infty\}$, is given by

$$\partial g(a) = \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a \rangle, \ \forall x \in \mathbb{R}^n\}, \tag{2.4}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product on $\mathbb{R}^n$.

(2) The ϵ-subdifferential of g at $a \in \operatorname{dom} g$ is given by

$$\partial_\epsilon g(a) = \{v \in \mathbb{R}^n \mid g(x) \geqq g(a) + \langle v, x - a \rangle - \epsilon, \ \forall x \in \mathbb{R}^n\}. \tag{2.5}$$

Definition 2.2. Let $\epsilon \geqq 0$. Then $\overline{x} \in A$ is called an ϵ-approximate solution of SDP, if, for any $x \in A$,

$$f(x) \geqq f(\overline{x}) - \epsilon. \tag{2.6}$$

Definition 2.3. The conjugate function of a function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$g^*(v) = \sup\{\langle v, x\rangle - g(x) \mid x \in \mathbb{R}^n\}. \tag{2.7}$$

Definition 2.4. The epigraph of a function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $epi\ g$, is defined by

$$epig = \{(x, r) \in \mathbb{R}^n \times \mathbb{R} \mid g(x) \leqq r\}. \tag{2.8}$$

If g is sublinear (i.e., convex and positively homogeneous of degree one), then $\partial_\epsilon g(0) = \partial g(0)$, for all $\epsilon \geqq 0$. If $\tilde{g}(x) = g(x) - k$, $x \in \mathbb{R}^n$, $k \in \mathbb{R}$, then $epi\tilde{g}^* = epig^* + (0, k)$. It is worth nothing that if g is sublinear, then

$$epig^* = \partial g(0) \times \mathbb{R}_+. \tag{2.9}$$

Moreover, if g is sublinear and if $\tilde{g}(x) = g(x) - k$, $x \in \mathbb{R}^n$, and $k \in \mathbb{R}$, then

$$epi\tilde{g}^* = \partial g(0) \times [k, \infty). \tag{2.10}$$

Definition 2.5. Let C be a closed convex set in $\mathbb{R}^n$ and $x \in C$.

(1) Let $N_C(x) = \{v \in \mathbb{R}^n \mid \langle v, y - x\rangle \leqq 0, \text{for all } y \in C\}$. Then $N_C(x)$ is called the normal cone to C at x.

(2) Let $\epsilon \geqq 0$. Let $N_C^\epsilon(x) = \{v \in \mathbb{R}^n \mid \langle v, y - x\rangle \leqq \epsilon, \text{for all } y \in C\}$. Then $N_C^\epsilon(x)$ is called the ϵ-normal set to C at x.

(3) When C is a closed convex cone in R^n, $N_C(0)$ we denoted by C^* and called the negative dual cone of C.

Proposition 2.6 (see [17, 18]). *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let δ_C be the indicator function with respect to a closed convex subset C of $\mathbb{R}^n$, that is, $\delta_C(x) = 0$ if $x \in C$, and $\delta_C(x) = +\infty$ if $x \notin C$. Let $\epsilon \geqq 0$. Then*

$$\partial_\epsilon(f + \delta_C)(\overline{x}) = \bigcup_{\substack{\epsilon_0 \geqq 0, \epsilon_1 \geqq 0 \\ \epsilon_0 + \epsilon_1 = \epsilon}} \{\partial_{\epsilon_0} f(\overline{x}) + \partial_{\epsilon_1} \delta_C(\overline{x})\}. \tag{2.11}$$

Proposition 2.7 (see [7]). *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a continuous convex function and let $h : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then*

$$epi(g + h)^* = epig^* + epih^*. \tag{2.12}$$

Following the proof of Lemma 2.2 in [1], we can prove the following lemma.

Lemma 2.8. *Let $F_i \in S_n, i = 0, 1, \ldots, m$. Suppose that $A \neq \emptyset$. Let $u \in \mathbb{R}^m$ and $\alpha \in \mathbb{R}$. Then the following are equivalent:*

$$\text{(i)} \quad \left\{ x \in C \mid F_0 + \sum_{i=1}^{m} F_i x_i \succeq 0 \right\} \subset \{ x \in \mathbb{R}^m \mid \langle u, x \rangle \geqq \alpha \},$$
$$\text{(ii)} \quad \begin{pmatrix} u \\ \alpha \end{pmatrix} \in \operatorname{cl}\left(\bigcup_{(Z,\delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \widehat{F}^*(Z) \\ -\operatorname{Tr}[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+ \right). \tag{2.13}$$

3. ϵ-Duality Theorem

Now we give ϵ-duality theorems for SDP. Using Lemma 2.8, we can obtain the following lemma which is useful in proving our ϵ-strong duality theorems for SDP.

Lemma 3.1. *Let $\overline{x} \in A$. Suppose that*

$$\bigcup_{(Z,\delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \widehat{F}^*(Z) \\ -\operatorname{Tr}[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+ \tag{3.1}$$

is closed. Then $\overline{x}$ is an ϵ-approximate solution of SDP if and only if there exists $Z \in S$ such that for any $x \in C$,

$$f(x) - \operatorname{Tr}[ZF(x)] \geqq f(\overline{x}) - \epsilon. \tag{3.2}$$

Proof. ($\Longrightarrow$) Let $\overline{x}$ be an ϵ-approximate solution of SDP. Then $f(x) \geqq f(\overline{x}) - \epsilon$, for any $x \in A$. Let $h(x) = f(x) - f(\overline{x}) + \epsilon$. Then $h(x) + \delta_A(x) \geqq 0$, for any $x \in \mathbb{R}^n$. Thus we have, from Proposition 2.7,

$$\begin{aligned} 0 \in epi(h + \delta_A)^* &= epih^* + epi\delta_A{}^* \\ &= epif^* + (0, f(\overline{x}) - \epsilon) + epi\delta_A^*, \end{aligned} \tag{3.3}$$

and hence, $(0, \epsilon - f(\overline{x})) \in epif^* + epi\delta_A^*$. So there exists $(u, r) \in epif^*$ such that $(-u, \epsilon - f(\overline{x}) - r) \in epi\delta_A{}^*$ and hence there exists $(u, r) \in epif^*$ such that $\langle -u, x \rangle \leqq \epsilon - f(\overline{x}) - r$ for any $x \in A$. Since $f^*(u) \leqq r$, $\langle -u, x \rangle \leqq \epsilon - f(\overline{x}) - f^*(u)$ for any $x \in A$; and hence it follows from Lemma 2.8 that

$$\begin{pmatrix} u \\ -\epsilon + f(\overline{x}) + f^*(u) \end{pmatrix} \in \bigcup_{(Z,\delta) \in S \times \mathbb{R}_+} \left\{ \begin{pmatrix} \widehat{F}^*(Z) \\ -\operatorname{Tr}[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+. \tag{3.4}$$

Thus there exist $(Z, \delta) \in S \times \mathbb{R}_+, c^* \in C^*$, and $\gamma \in \mathbb{R}_+$ such that

$$\begin{aligned} u &= \widehat{F}^*(Z) - c^*, \\ -\epsilon + f(\overline{x}) + f^*(u) &= -\operatorname{Tr}[ZF_0] - \delta - \gamma. \end{aligned} \tag{3.5}$$

This gives

$$\begin{aligned}\left\langle \widehat{F}^*(Z), x\right\rangle - \langle c^*, x\rangle - f(x) = \langle u, x\rangle - f(x) &\leqq f^*(u)\\ &= -\operatorname{Tr}[ZF_0] - \delta - \gamma - f(\overline{x}) + \epsilon,\end{aligned} \tag{3.6}$$

for any $x \in \mathbb{R}^n$. Thus we have

$$\begin{aligned} f(\overline{x}) - \epsilon &\leqq -\langle u, x\rangle + f(x) - \operatorname{Tr}[ZF_0] - \delta - \gamma \\ &= f(x) - \left\langle \widehat{F}^*(Z), x\right\rangle + \langle c^*, x\rangle - \operatorname{Tr}[ZF_0] - \delta - \gamma \\ &= f(x) - \operatorname{Tr}[ZF(x)] + \langle c^*, x\rangle - \delta - \gamma \\ &\leqq f(x) - \operatorname{Tr}[ZF(x)] \end{aligned} \tag{3.7}$$

for any $x \in C$.

($\Longleftarrow$) Suppose that there exists $Z \in S$ such that

$$f(x) - \operatorname{Tr}[ZF(x)] \geqq f(\overline{x}) - \epsilon, \tag{3.8}$$

for any $x \in C$. Then we have

$$f(x) \geqq f(x) - \operatorname{Tr}[ZF(x)] \geqq f(\overline{x}) - \epsilon, \tag{3.9}$$

for any $x \in A$. Thus $f(x) \geqq f(\overline{x}) - \epsilon$, for any $x \in A$. Hence $\overline{x}$ is an ϵ-approximate solution of SDP. □

Now we formulate the dual problem SDD of SDP as follows:

$$\begin{aligned} \text{(SDD)} \quad \text{maximize} \quad & f(x) - \operatorname{Tr}[ZF(x)], \\ \text{subject to} \quad & 0 \in \partial_{\epsilon_0} f(x) - \widehat{F}^*(Z) + N_C^{\epsilon_1}(x), \\ & Z \succeq 0, \\ & \epsilon_0 + \epsilon_1 \in [0, \epsilon]. \end{aligned} \tag{3.10}$$

We prove ϵ-weak and ϵ-strong duality theorems which hold between SDP and SDD.

Theorem 3.2 (ϵ-weak duality). *For any feasible solution x of SDP and any feasible solution (y, Z) of SDD,*

$$f(x) \geqq f(y) - \operatorname{Tr}[ZF(y)] - \epsilon. \tag{3.11}$$

Proof. Let x and (y, Z) be feasible solutions of SDP and SDD respectively. Then $\mathrm{Tr}[ZF(x)] \geqq 0$ and there exist $v \in \partial_{\epsilon_0} f(y)$ and $\omega \in N_C^{\epsilon_1}(y)$ such that $v = -\omega + \widehat{F}^*(Z)$. Thus, we have

$$\begin{aligned}
f(x) - \{f(y) - \mathrm{Tr}[ZF(y)]\} &\geqq \langle v, x - y\rangle - \epsilon_0 + \mathrm{Tr}[ZF(y)] \\
&= \left\langle -\omega + \widehat{F}^*(Z), x - y \right\rangle - \epsilon_0 + \mathrm{Tr}[ZF(y)] \\
&\geqq \left\langle \widehat{F}^*(Z), x - y \right\rangle - \epsilon_0 - \epsilon_1 + \mathrm{Tr}[ZF(y)] \\
&= \left\langle \widehat{F}^*(Z), x \right\rangle - \left\langle \widehat{F}^*(Z), y \right\rangle - \epsilon_0 - \epsilon_1 \\
&\quad + \mathrm{Tr}[ZF(y)] \\
&= \mathrm{Tr}\left[Z\sum_{i=1}^{m} x_i F_i\right] - \mathrm{Tr}\left[Z\sum_{i=1}^{m} y_i F_i\right] - \epsilon_0 - \epsilon_1 \\
&\quad + \mathrm{Tr}[ZF_0] + \mathrm{Tr}\left[Z\sum_{i=1}^{m} y_i F_i\right] \\
&= \mathrm{Tr}[ZF(x)] - \epsilon_0 - \epsilon_1 \\
&\geqq -\epsilon_0 - \epsilon_1 \\
&\geqq -\epsilon.
\end{aligned} \tag{3.12}$$

Hence $f(x) \geqq f(y) - \mathrm{Tr}[ZF(y)] - \epsilon$. □

Theorem 3.3 (ϵ-strong duality). *Suppose that*

$$\bigcup_{(Z,\delta)\in S\times\mathbb{R}_+} \left\{ \begin{pmatrix} \widehat{F}^*(Z) \\ -\mathrm{Tr}[ZF_0] - \delta \end{pmatrix} \right\} - C^* \times \mathbb{R}_+ \tag{3.13}$$

is closed. If $\overline{x}$ is an ϵ-approximate solution of SDP, then there exists $\overline{Z} \in S$ such that $(\overline{x}, \overline{Z})$ is a 2ϵ-approximate solution of SDD.

Proof. Let $\overline{x} \in A$ be an ϵ-approximate solution of SDP. Then $f(x) \geqq f(\overline{x}) - \epsilon$, for any $x \in A$. By Lemma 3.1, there exists $\overline{Z} \in S$ such that

$$f(x) - \mathrm{Tr}\left[\overline{Z}F(x)\right] \geqq f(\overline{x}) - \epsilon, \tag{3.14}$$

for any $x \in C$. Letting $x = \overline{x}$ in (3.14), $\mathrm{Tr}[\overline{Z}F(\overline{x})] \leqq \epsilon$. Since $F(\overline{x}) \in S$ and $\overline{Z} \in S$, $\mathrm{Tr}[\overline{Z}F(\overline{x})] \geqq 0$.

Thus from (3.14),

$$f(x) - \mathrm{Tr}\left[\overline{Z}F(x)\right] + \epsilon \geqq f(\overline{x}) - \mathrm{Tr}\left[\overline{Z}F(\overline{x})\right] \tag{3.15}$$

for any $x \in C$. Hence $\overline{x}$ is an ϵ-approximate solution of the following problem:

$$\begin{aligned} &\text{maximize} \quad f(x) - \operatorname{Tr}\left[\overline{Z}F(x)\right], \\ &\text{subject to} \quad x \in C, \end{aligned} \tag{3.16}$$

and so, $0 \in \partial_{\epsilon}(f - \widehat{F}^*(\overline{Z}) + \delta_C)\ (\overline{x})$, and hence, by Proposition 2.6, there exist ϵ_0, $\epsilon_1 \in [0, \epsilon]$ such that $\epsilon_0 + \epsilon_1 = \epsilon$ and

$$0 \in \partial_{\epsilon_0} f(\overline{x}) - \widehat{F}^*\left(\overline{Z}\right) + N_C^{\epsilon_1}(\overline{x}). \tag{3.17}$$

So, $(\overline{x}, \overline{Z})$ is a feasible solution of SDD. For any feasible solution (y, Z) of SDD,

$$\begin{aligned} f(\overline{x}) - \operatorname{Tr}\left[\overline{Z}F(\overline{x})\right] - \{f(y) - \operatorname{Tr}[ZF(y)]\} &= f(\overline{x}) - \{f(y) - \operatorname{Tr}[ZF(y)]\} \\ &\quad - \operatorname{Tr}\left[\overline{Z}F(\overline{x})\right] \\ &\geqq -\epsilon - \operatorname{Tr}\left[\overline{Z}F(\overline{x})\right] \\ &\quad (\text{by } \epsilon\text{-weak duality}) \\ &\geqq -\epsilon - \epsilon \\ &= -2\epsilon. \end{aligned} \tag{3.18}$$

Thus $(\overline{x}, \overline{Z})$ is a 2ϵ-approximate solution to SDD. □

Now we characterize the ϵ-normal set to $\mathbb{R}^n_+$.

Proposition 3.4. *Let* $(x_1, \ldots, x_n) \in \mathbb{R}^n_+$ *and* $\epsilon \geqq 0$. *Then*

$$N^{\epsilon}_{\mathbb{R}^n_+}(x_1, \ldots, x_n) = \bigcup_{\substack{\epsilon_i \geqq 0 \\ \sum_{i=1}^n \epsilon_i = \epsilon}} \prod_{i=1}^{n} A(\epsilon_i), \tag{3.19}$$

where

$$A(\epsilon_i) = \begin{cases} -\mathbb{R}_+ & \text{if } x_i = 0, \\ \left[-\dfrac{\epsilon_i}{x_i}, 0\right] & \text{if } x_i > 0. \end{cases} \tag{3.20}$$

Proof. Let $(x_1, \dots, x_n) \in \mathbb{R}^n_+$ and $\epsilon \geqq 0$. Then

$$\begin{aligned}
N^{\epsilon}_{\mathbb{R}^n_+}(x_1, \dots, x_n) &= \partial_\epsilon \delta_{\mathbb{R}^n_+}(x_1, \dots, x_n) \\
&= \partial_\epsilon \left(\sum_{i=1}^{n} \delta_{\mathbb{R}\times\cdots\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\cdots\times\mathbb{R}} \right)(x_1, \dots, x_n) \\
&= \bigcup_{\substack{\epsilon_i \geqq 0 \\ \sum_{i=1}^n \epsilon_i = \epsilon}} \sum_{i=1}^{n} \partial_{\epsilon_i} \delta_{\mathbb{R}\times\cdots\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\cdots\times\mathbb{R}}(x_1, \dots, x_n).
\end{aligned} \tag{3.21}$$

Let $(v_1, \dots, v_n) \in \partial_{\epsilon_i} \delta_{\mathbb{R}\times\cdots\times\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}\times\cdots\times\mathbb{R}}(x_1, \dots, x_n)$ (where $\mathbb{R}_+$ is at the ith position in $\mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times \cdots \times \mathbb{R}$)

$$\begin{aligned}
\Longleftrightarrow \quad & \text{for any } (y_1, \dots, y_n) \in \mathbb{R} \times \cdots \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R} \times \cdots \times \mathbb{R}, \\
& \epsilon_i \geqq v_1(y_1 - x_1) + \cdots + v_i(y_i - x_i) + \cdots + v_n(y_n - x_n), \\
\Longleftrightarrow \quad & \text{for any } y_i \in \mathbb{R}_+, \quad \epsilon_i \geqq v_i(y_i - x_i), \quad v_j = 0, \\
& \text{for } j \in \{1, \dots, n\} \setminus \{i\}, \\
\Longleftrightarrow \quad & v_i \in \begin{cases} -\mathbb{R}_+, & \text{if } x_i = 0, \\ \left[-\dfrac{\epsilon_i}{x_i}, 0 \right], & \text{if } x_i > 0, \end{cases} \\
& v_j = 0 \text{ for } j \in \{1, \dots, n\} \setminus \{i\}.
\end{aligned} \tag{3.22}$$

Thus, we have

$$\begin{aligned}
N^{\epsilon_1}_{\mathbb{R}^n_+}(x_1, \dots, x_n) &= \bigcup_{\substack{\epsilon_i \geqq 0 \\ \sum_{i=1}^n \epsilon_i = \epsilon}} \sum_{i=1}^{n} \{0\} \times \cdots \times \{0\} \times A(\epsilon_i) \times \{0\} \times \cdots \times \{0\} \\
&= \bigcup_{\substack{\epsilon_i \geqq 0 \\ \sum_{i=1}^n \epsilon_i = \epsilon}} \prod_{i=1}^{n} A(\epsilon_i).
\end{aligned} \tag{3.23}$$

$\square$

From Proposition 3.4, we can calculate $N^{\epsilon}_{\mathbb{R}^2_+}$.

Corollary 3.5. *Let* $(x_1, x_2) \in \mathbb{R}^2_+$ *and* $\epsilon \geqq 0$. *Then following hold.*

(i) *If* $(x_1, x_2) = (0, 0)$, *then* $N^{\epsilon}_{\mathbb{R}^2_+}(0, 0) = -\mathbb{R}^2_+$.

(ii) *If* $(x_1, x_2) = (x_1, 0)$ *and* $x_1 > 0$, *then* $N^{\epsilon}_{\mathbb{R}^2_+}(x_1, 0) = [-\epsilon/x_1, 0] \times (-\infty, 0]$.

(iii) *If* $(x_1, x_2) = (0, x_2)$ *and* $x_2 > 0$, *then* $N^{\epsilon}_{\mathbb{R}^2_+}(0, x_2) = (-\infty, 0] \times [-\epsilon/x_2, 0]$.

(iv) *If* $(x_1, x_2) = (x_1, x_2)$, *and* $x_1 > 0$ *and* $x_2 > 0$, *then*

$$N^{\epsilon}_{\mathbb{R}^2_+}(x_1, x_2) = \bigcup_{\substack{\epsilon_1 \geqq 0, \epsilon_2 \geqq 0, \\ \epsilon_1 + \epsilon_2 = \epsilon}} \left[-\frac{\epsilon_1}{x_1}, 0\right] \times \left[-\frac{\epsilon_2}{x_2}, 0\right]. \tag{3.24}$$

Now we give an example illustrating our ϵ-duality theorems.

Example 3.6. Consider the following convex semidefinite program.

$$\begin{aligned} \text{(SDP)} \qquad & \text{Minimize} \quad x_1 + x_2^2, \\ & \text{subject to} \quad \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix} \succeq 0, \\ & \qquad\qquad (x_1, x_2) \in \mathbb{R}^2{}_+. \end{aligned} \tag{3.25}$$

Let $f(x_1, x_2) = x_1 + x_2^2$,

$$F_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad F_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad F_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{3.26}$$

and $\epsilon \geqq 0$. Let $f(x_1, x_2) = x_1 + x_2^2$ and

$$F(x_1, x_2) = \begin{pmatrix} 0 & x_1 \\ x_1 & 0 \end{pmatrix}. \tag{3.27}$$

Then $A := \{(0, x_2) \in \mathbb{R}^2 \mid x_2 \geqq 0\}$ is the set of all feasible solutions of SDP and the set of all ϵ-approximate solutions of SDP is $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0, 0 \leqq x_2 \leqq \sqrt{\epsilon}\}$. Let $F = \{((x_1, x_2), Z) \mid 0 \in \partial_{\epsilon_0} f(x_1, x_2) - \widehat{F}^*(Z) + N^{\epsilon_1}_{\mathbb{R}^2_+}(x_1, x_2), Z \succeq 0, \epsilon_0 + \epsilon_1 \in [0, \epsilon]\}$. Then F is the set of all feasible solution of SDD. Now we calculate the set F.

$$\widetilde{A} := \left\{ \left((0,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid 0 \in \partial_{\epsilon_0} f(0,0) - \widehat{F}^* \begin{pmatrix} a & b \\ b & c \end{pmatrix} + N_{\mathbb{R}^2_+}^{\epsilon_1}(0,0), \right.$$

$$\left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\}$$

$$= \left\{ \left((0,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid 0 \in \{1\} \times \left[-2\sqrt{\epsilon_0}, 2\sqrt{\epsilon_0}\right] - (2b,0) - \mathbb{R}^2_+, \right.$$

$$\left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\}$$

$$= \left\{ \left((0,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid (2b,0) \in (-\infty,1] \times \left(-\infty, 2\sqrt{\epsilon_0}\right], \right.$$

$$\left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\}$$

$$= \left\{ \left((0,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid a \geqq 0,\ c \geqq 0,\ b \leqq \frac{1}{2},\ b^2 \leqq ac \right\},$$

$$\widetilde{B} := \left\{ \left((0,x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_2 > 0,\ 0 \in \partial_{\epsilon_0} f(0,x_2) - \widehat{F}^* \begin{pmatrix} a & b \\ b & c \end{pmatrix} + N_{\mathbb{R}^2_+}^{\epsilon_1}(0,x_2) \right.$$

$$\left. a \geqq 0,\quad c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\}$$

$$= \left\{ \left((0,x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_2 > 0,\ 0 \in \{1\} \times \left[2x_2 - 2\sqrt{\epsilon_0}, 2x_2 + 2\sqrt{\epsilon_0}\right] - (2b,0) \right.$$

$$\left. + (-\infty,0] \times \left[-\frac{\epsilon_1}{x_2}, 0\right],\quad a \geqq 0,\quad c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\}$$

$$= \left\{ \left((0,x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_2 > 0,\ (2b,0) \in (-\infty,1] \times \left[2x_2 - \frac{\epsilon_1}{x_2} - 2\sqrt{\epsilon_0}, 2x_2 + 2\sqrt{\epsilon_0}\right], \right.$$

$$\left. a \geqq 0,\ c \geqq 0,\quad b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\}$$

$$= \left\{ \left((0,x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid 0 < x_2 \leqq \frac{\sqrt{\epsilon_0} + \sqrt{\epsilon_0 + 2\epsilon_1}}{2},\ a \geqq 0,\ c \geqq 0,\ b \leqq \frac{1}{2},\ b^2 \leqq ac, \right.$$

$$\left. \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\},$$

$$\widetilde{C} := \left\{ \left((x_1,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0,\ 0 \in \partial_{\epsilon_0} f(x_1,0) - \widehat{F}^* \begin{pmatrix} a & b \\ b & c \end{pmatrix} + N^{\epsilon_1}_{\mathbb{R}^2_+}(x_1,0), \right.$$

$$\left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\}$$

$$= \left\{ \left((x_1,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0,\ 0 \in \{1\} \times \left[-2\sqrt{\epsilon_0}, +2\sqrt{\epsilon_0} \right] - (2b,0) \right.$$

$$\left. + \left[-\frac{\epsilon_1}{x_1}, 0 \right] \times (-\infty, 0],\ a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\}$$

$$= \left\{ \left((x_1,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0,\ (2b,0) \in \left[1 - \frac{\epsilon_1}{x_1}, 1 \right] \times \left(-\infty, 2\sqrt{\epsilon_0} \right], \right.$$

$$\left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\}$$

$$= \left\{ \left((x_1,0), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid 0 < x_1,\ -\epsilon_1 \leqq -x_1 + 2bx_1,\ a \geqq 0,\ c \geqq 0,\ b \leqq \frac{1}{2},\ b^2 \leqq ac, \right.$$

$$\left. \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\},$$

$$\widetilde{D} := \left\{ \left((x_1,x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0,\ x_2 > 0,\ 0 \in \partial_{\epsilon_0} f(x_1,x_2) - \widehat{F}^* \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right.$$

$$\left. + N^{\epsilon_1}_{\mathbb{R}^2_+}(x_1,x_2),\ a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\}$$

$$= \left\{ \left((x_1,x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0, x_2 > 0, \right.$$

$$0 \in \{1\} \times \left[2x_2 - 2\sqrt{\epsilon_0}, 2x_2 + 2\sqrt{\epsilon_0} \right] - (2b,0) + \left[-\frac{\epsilon_1^1}{x_1}, 0 \right] \times \left[-\frac{\epsilon_1^2}{x_2}, 0 \right],$$

$$\left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_1^1 + \epsilon_1^2 = \epsilon_1,\ \epsilon_0 + \epsilon_1 \in [0,\ \epsilon] \right\}$$

$$= \left\{ \left((x_1,x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid x_1 > 0,\ x_2 > 0, \right.$$

$$(2b,0) \in \left[1 - \frac{\epsilon_1^1}{x_1}, 1 \right] \times \left[2x_2 - \frac{\epsilon_1^2}{x_2} - 2\sqrt{\epsilon_0}, 2x_2 + 2\sqrt{\epsilon_0} \right],$$

$$\left. a \geqq 0,\ c \geqq 0,\ b^2 \leqq ac,\ \epsilon_1^1 + \epsilon_1^2 = \epsilon_1,\ \epsilon_0 + \epsilon_1 \in [0,\epsilon] \right\}$$

$$= \left\{ \left((x_1, x_2), \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right) \mid 0 < x_1,\ -\epsilon_1^1 \leqq -x_1 + 2bx_1, \right.$$

$$0 < x_2 \leqq \frac{\sqrt{\epsilon_0} + \sqrt{\epsilon_0 + 2\epsilon_1^2}}{2},\ a \geqq 0,\ c \geqq 0,\ b \leqq \frac{1}{2},\ b^2 \leqq ac,\ \epsilon_1^1 + \epsilon_1^2 = \epsilon_1,$$

$$\left. \epsilon_0 + \epsilon_1 \in [0, \epsilon] \right\}. \tag{3.28}$$

Thus $F = \widetilde{A} \cup \widetilde{B} \cup \widetilde{C} \cup \widetilde{D}$. We can check that for any $(x_1, x_2) \in A$ and any $((y_1, y_2), \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)) \in F$,

$$f(x_1, x_2) \geqq f(y_1, y_2) - \operatorname{Tr}\left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} F(y_1, y_2) \right) - \epsilon, \tag{3.29}$$

that is, ϵ-weak duality holds.

Let $(\overline{x}_1, \overline{x}_2) \in A$ be an ϵ-approximate solution of SDP. Then $\overline{x}_1 = 0$ and $0 \leqq \overline{x}_2 \leqq \sqrt{\epsilon}$. So, we can easily check that $((\overline{x}_1, \overline{x}_2), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)) \in F$.

Since $\operatorname{Tr}(\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) F(\overline{x}_1, \overline{x}_2))) = 0$, from (3.29),

$$f(\overline{x}_1, \overline{x}_2) \geqq f(y_1, y_2) - \operatorname{Tr}\left(\begin{pmatrix} a & b \\ b & c \end{pmatrix} F(y_1, y_2) \right) - \epsilon, \tag{3.30}$$

for any $((y_1, y_2), \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)) \in F$. So $((\overline{x}_1, \overline{x}_2), \left(\begin{smallmatrix} \overline{a} & \overline{b} \\ \overline{b} & \overline{c} \end{smallmatrix}\right))$ is an ϵ-approximate solution of SDD. Hence ϵ-strong duality holds.

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